KUMAR CLASSES

Excellence in Mathematics

CLASS 11 MATHEMATICS

Chapter 8 : SEQUENCES AND SERIES – NCERT SOLUTIONS

Complete Concept Notes

Full Concepts · Solved Examples · Board Questions · Concept Check

Strong Concepts + Regular Practice = High Score

Exercise 8.1

Q1. Write the first five terms of the sequence whose n^{th} term is $a_n = n(n+2)$.

Solution :

- Put $n = 1, 2, 3, 4, 5$ turn by turn in $a_n = n(n+2)$.
- $a_1 = 1(1+2) = 3, \quad a_2 = 2(2+2) = 8, \quad a_3 = 3(3+2) = 15$
- $a_4 = 4(4+2) = 24, \quad a_5 = 5(5+2) = 35$

Answer : $3,\ 8,\ 15,\ 24,\ 35$

Q2. Write the first five terms of the sequence whose n^{th} term is $a_n = \dfrac{n}{n+1}$.

Solution :

- Put $n = 1, 2, 3, 4, 5$ in $a_n = \dfrac{n}{n+1}$.
- $a_1 = \dfrac{1}{2},\ a_2 = \dfrac{2}{3},\ a_3 = \dfrac{3}{4},\ a_4 = \dfrac{4}{5},\ a_5 = \dfrac{5}{6}$

Answer : $\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \dfrac{5}{6}$

Q3. Write the first five terms of the sequence whose n^{th} term is $a_n = 2^n$.

Solution :

- Put $n = 1, 2, 3, 4, 5$ in $a_n = 2^n$.
- $a_1 = 2,\ a_2 = 4,\ a_3 = 8,\ a_4 = 16,\ a_5 = 32$

Answer : $2,\ 4,\ 8,\ 16,\ 32$

Q4. Write the first five terms of the sequence whose n^{th} term is $a_n = \dfrac{2n-3}{6}$.

Solution :

- Put $n = 1, 2, 3, 4, 5$ in $a_n = \dfrac{2n-3}{6}$.
- $a_1 = \dfrac{-1}{6},\ a_2 = \dfrac{1}{6},\ a_3 = \dfrac{3}{6} = \dfrac{1}{2},\ a_4 = \dfrac{5}{6},\ a_5 = \dfrac{7}{6}$

Answer : $-\dfrac{1}{6}, \dfrac{1}{6}, \dfrac{1}{2}, \dfrac{5}{6}, \dfrac{7}{6}$

Q5. Write the first five terms of the sequence whose n^{th} term is $a_n = (-1)^{n-1} 5^{n+1}$.

Solution :

- $a_1 = (-1)^0 5^2 = 25, \quad a_2 = (-1)^1 5^3 = -125$
- $a_3 = (-1)^2 5^4 = 625, \quad a_4 = (-1)^3 5^5 = -3125$
- $a_5 = (-1)^4 5^6 = 15625$

Answer : $25,\ -125,\ 625,\ -3125,\ 15625$

Q6. Write the first five terms of the sequence whose n^{th} term is $a_n = n\,\dfrac{n^2+5}{4}$.

Solution :

- $a_1 = 1 \cdot \dfrac{1+5}{4} = \dfrac{6}{4} = \dfrac{3}{2}, \quad a_2 = 2 \cdot \dfrac{4+5}{4} = \dfrac{18}{4} = \dfrac{9}{2}$
- $a_3 = 3 \cdot \dfrac{9+5}{4} = \dfrac{42}{4} = \dfrac{21}{2}, \quad a_4 = 4 \cdot \dfrac{16+5}{4} = 21$
- $a_5 = 5 \cdot \dfrac{25+5}{4} = \dfrac{150}{4} = \dfrac{75}{2}$

Answer : $\dfrac{3}{2}, \dfrac{9}{2}, \dfrac{21}{2}, 21, \dfrac{75}{2}$

Q7. Find the indicated terms a_{17} and a_{24} of the sequence whose n^{th} term is $a_n = 4n - 3$.

Solution :

- $a_{17} = 4(17) - 3 = 68 - 3 = 65$
- $a_{24} = 4(24) - 3 = 96 - 3 = 93$

Answer : $a_{17} = 65, \;\; a_{24} = 93$

Q8. Find the indicated term a_7 of the sequence whose n^{th} term is $a_n = \dfrac{n^2}{2^n}$.

Solution :

- Put $n = 7$ in $a_n = \dfrac{n^2}{2^n}$.

- $a_7 = \dfrac{7^2}{2^7} = \dfrac{49}{128}$

Answer : $a_7 = \dfrac{49}{128}$

Q9. Find the indicated term a_9 of the sequence whose n^{th} term is $a_n = (-1)^{n-1} n^3$.

Solution :

- Put $n = 9$; here $(-1)^{9-1} = (-1)^8 = 1$.
- $a_9 = (1)(9)^3 = 729$

Answer : $a_9 = 729$

Q10. Find the indicated term a_{20} of the sequence whose n^{th} term is $a_n = \dfrac{n(n-2)}{n+3}$.

Solution :

- Put $n = 20$ in $a_n = \dfrac{n(n-2)}{n+3}$.
- $a_{20} = \dfrac{20(20-2)}{20+3} = \dfrac{20 \times 18}{23} = \dfrac{360}{23}$

Answer : $a_{20} = \dfrac{360}{23}$

Q11. Write the first five terms and the corresponding series for $a_1 = 3,\ a_n = 3a_{n-1} + 2$ for all $n > 1$.

Solution :

- $a_1 = 3$
- $a_2 = 3a_1 + 2 = 3(3) + 2 = 11$
- $a_3 = 3a_2 + 2 = 3(11) + 2 = 35$
- $a_4 = 3a_3 + 2 = 3(35) + 2 = 107$
- $a_5 = 3a_4 + 2 = 3(107) + 2 = 323$
- *Corresponding series:* $3 + 11 + 35 + 107 + 323 + \cdots$

Answer : $3,\ 11,\ 35,\ 107,\ 323;\quad 3 + 11 + 35 + 107 + 323 + \cdots$

Q12. Write the first five terms and the corresponding series for $a_1 = -1,\ a_n = \dfrac{a_{n-1}}{n},\ n \geq 2.$

Solution :

- $a_1 = -1$
- $a_2 = \dfrac{a_1}{2} = -\dfrac{1}{2}, \quad a_3 = \dfrac{a_2}{3} = -\dfrac{1}{6}$
- $a_4 = \dfrac{a_3}{4} = -\dfrac{1}{24}, \quad a_5 = \dfrac{a_4}{5} = -\dfrac{1}{120}$
- *Corresponding series:* $-1 - \dfrac{1}{2} - \dfrac{1}{6} - \dfrac{1}{24} - \dfrac{1}{120} - \cdots$

Answer : $-1,\ -\dfrac{1}{2},\ -\dfrac{1}{6},\ -\dfrac{1}{24},\ -\dfrac{1}{120}$

Q13. Write the first five terms and the corresponding series for $a_1 = a_2 = 2,\ a_n = a_{n-1} - 1,\ n > 2.$

Solution :

- $a_1 = 2,\ a_2 = 2$
- $a_3 = a_2 - 1 = 1, \quad a_4 = a_3 - 1 = 0, \quad a_5 = a_4 - 1 = -1$
- *Corresponding series:* $2 + 2 + 1 + 0 + (-1) + \cdots$

Answer : $2,\ 2,\ 1,\ 0,\ -1; \quad 2+2+1+0-1+\cdots$

Q14. The Fibonacci sequence is defined by $a_1 = a_2 = 1$ and $a_n = a_{n-1} + a_{n-2},\ n > 2$. Find $\dfrac{a_{n+1}}{a_n}$ for $n = 1, 2, 3, 4, 5$.

Solution :

- *First find the terms:* $a_1 = 1,\ a_2 = 1,\ a_3 = 2,\ a_4 = 3,\ a_5 = 5,\ a_6 = 8$.
- $n = 1 : \ \dfrac{a_2}{a_1} = \dfrac{1}{1} = 1$
- $n = 2 : \ \dfrac{a_3}{a_2} = \dfrac{2}{1} = 2$
- $n = 3 : \ \dfrac{a_4}{a_3} = \dfrac{3}{2}$
- $n = 4 : \ \dfrac{a_5}{a_4} = \dfrac{5}{3}$
- $n = 5 : \ \dfrac{a_6}{a_5} = \dfrac{8}{5}$

Answer : $1,\ 2,\ \dfrac{3}{2},\ \dfrac{5}{3},\ \dfrac{8}{5}$

Exercise 8.2

Q1. Find the 20^{th} and n^{th} terms of the G.P. $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \ldots$

Solution :

- *Here* $a = \frac{5}{2}$ *and* $r = \frac{5/4}{5/2} = \frac{1}{2}$.
- $a_n = ar^{n-1} = \frac{5}{2}\left(\frac{1}{2}\right)^{n-1} = \frac{5}{2^n}$
- $a_{20} = \frac{5}{2^{20}} = \frac{5}{1048576}$

Answer : $a_n = \frac{5}{2^n}$, $a_{20} = \frac{5}{2^{20}}$

Q2. Find the 12^{th} term of a G.P. whose 8^{th} term is 192 and the common ratio is 2.

Solution :

- $a_8 = ar^7 = 192 \Rightarrow a(2)^7 = 192 \Rightarrow 128a = 192 \Rightarrow a = \frac{3}{2}$.
- $a_{12} = ar^{11} = \frac{3}{2}(2)^{11} = \frac{3}{2} \times 2048 = 3072$
- *(Shortcut:* $a_{12} = a_8 \cdot r^4 = 192 \times 16 = 3072$.*)*

Answer : $a_{12} = 3072$

Q3. The 5^{th}, 8^{th} and 11^{th} terms of a G.P. are p, q and s respectively. Show that $q^2 = ps$.

Solution :

- $p = ar^4, \quad q = ar^7, \quad s = ar^{10}.$
- $ps = (ar^4)(ar^{10}) = a^2r^{14}$
- $q^2 = (ar^7)^2 = a^2r^{14}$
- $\therefore\ q^2 = ps$

Answer : $q^2 = ps$ (proved)

Q4. The 4^{th} term of a G.P. is the square of its second term, and the first term is -3. Determine its 7^{th} term.

Solution :

- $a = -3$. Given $a_4 = (a_2)^2 \Rightarrow ar^3 = (ar)^2 = a^2r^2$.
- Divide by ar^2: $r = a = -3$.
- $a_7 = ar^6 = (-3)(-3)^6 = (-3)(729) = -2187$

Answer : $a_7 = -2187$

Q5. Which term of the following sequences: (a) $2, 2\sqrt{2}, 4, \ldots$ is 128? (b) $\sqrt{3}, 3, 3\sqrt{3}, \ldots$ is 729? (c) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ is $\frac{1}{19683}$?

Solution :

- (a) $a = 2,\ r = \sqrt{2}$.
  $ar^{n-1} = 128 \Rightarrow 2(\sqrt{2})^{n-1} = 128 \Rightarrow (\sqrt{2})^{n-1} = 64 = 2^6$.
- $2^{(n-1)/2} = 2^6 \Rightarrow \frac{n-1}{2} = 6 \Rightarrow n = 13$.
- (b) $a = \sqrt{3},\ r = \sqrt{3}$. $(\sqrt{3})^n = 729 = 3^6 = (\sqrt{3})^{12} \Rightarrow n = 12$.
- (c) $a = \frac{1}{3},\ r = \frac{1}{3}$. $\left(\frac{1}{3}\right)^n = \frac{1}{19683} = \left(\frac{1}{3}\right)^9 \Rightarrow n = 9$.

Answer : (a) 13^{th},\ (b) 12^{th},\ (c) 9^{th} term

Q6. For what values of x are the numbers $-\frac{2}{7},\ x,\ -\frac{7}{2}$ in G.P.?

Solution :

- For G.P., middle term squared = product of the other two: $x^2 = \left(-\frac{2}{7}\right)\left(-\frac{7}{2}\right)$
  .
- $x^2 = 1\ \Rightarrow\ x = \pm 1$

Answer : $x = \pm 1$

Q7. Find the sum to 20 terms of the G.P.
$0.15,\ 0.015,\ 0.0015,\ \ldots$

Solution :

- $a = 0.15,\ r = \dfrac{0.015}{0.15} = 0.1.$
- $S_{20} = \dfrac{a(1-r^{20})}{1-r} = \dfrac{0.15\,(1-(0.1)^{20})}{1-0.1} = \dfrac{0.15}{0.9}(1-10^{-20})$
- $S_{20} = \dfrac{1}{6}(1-10^{-20})$

Answer : $S_{20} = \dfrac{1}{6}(1-10^{-20})$

Q8. Find the sum to n terms of the G.P. $\sqrt{7},\ \sqrt{21},\ 3\sqrt{7},\ \ldots$

Solution :

- $a = \sqrt{7},\ r = \dfrac{\sqrt{21}}{\sqrt{7}} = \sqrt{3}.$
- $S_n = \dfrac{a(r^n-1)}{r-1} = \dfrac{\sqrt{7}((\sqrt{3})^n-1)}{\sqrt{3}-1}$
- *Rationalise (multiply by $\sqrt{3}+1$):* $S_n = \dfrac{\sqrt{7}(\sqrt{3}+1)}{2}((\sqrt{3})^n-1)$

Answer : $S_n = \dfrac{\sqrt{7}(\sqrt{3}+1)}{2}\left((\sqrt{3})^n-1\right)$

Q9. Find the sum to n terms of the G.P.
$1,\ -a,\ a^2,\ -a^3,\ \ldots\ (a \neq -1).$

Solution :

- *First term* $= 1$, *common ratio* $r = -a.$

- $S_n = \dfrac{1-(-a)^n}{1-(-a)} = \dfrac{1-(-a)^n}{1+a}$

Answer : $S_n = \dfrac{1-(-a)^n}{1+a}$

Q10. Find the sum to n terms of the G.P. $x^3,\ x^5,\ x^7,\ \ldots\ (x \neq \pm 1)$.

Solution :

- $a = x^3,\ r = x^2.$
- $S_n = \dfrac{a(r^n-1)}{r-1} = \dfrac{x^3\left((x^2)^n-1\right)}{x^2-1} = \dfrac{x^3(x^{2n}-1)}{x^2-1}$

Answer : $S_n = \dfrac{x^3\left(x^{2n}-1\right)}{x^2-1}$

Q11. Evaluate $\sum_{k=1}^{11}(2+3^k)$.

Solution :

- $\sum_{k=1}^{11}(2+3^k) = \sum_{k=1}^{11}2 + \sum_{k=1}^{11}3^k = 22 + \left(3+3^2+\cdots+3^{11}\right)$
- $\sum_{k=1}^{11}3^k = \dfrac{3(3^{11}-1)}{3-1} = \dfrac{3(177147-1)}{2} = 265719$
- *Total* $= 22 + 265719 = 265741$

Answer : 265741

Q12. The sum of the first three terms of a G.P. is $\frac{39}{10}$ and their product is 1. Find the common ratio and the terms.

Solution :

- Take the three terms as $\frac{a}{r}$, a, ar. Product $= a^3 = 1 \Rightarrow a = 1$.
- Sum: $\frac{1}{r} + 1 + r = \frac{39}{10} \Rightarrow \frac{1}{r} + r = \frac{29}{10}$.
- $10r^2 - 29r + 10 = 0 \Rightarrow r = \frac{29 \pm 21}{20} \Rightarrow r = \frac{5}{2}$ or $\frac{2}{5}$.
- Terms: $\frac{2}{5}$, 1, $\frac{5}{2}$ (in either order).

Answer : $r = \frac{5}{2}$ or $\frac{2}{5}$; terms $\frac{2}{5}, 1, \frac{5}{2}$

Q13. How many terms of the G.P. $3,\ 3^2,\ 3^3,\ \ldots$ are needed to give the sum 120?

Solution :

- $a = 3,\ r = 3$. $S_n = \frac{a(r^n - 1)}{r - 1} = \frac{3(3^n - 1)}{2} = 120$.
- $3^n - 1 = 80 \Rightarrow 3^n = 81 = 3^4 \Rightarrow n = 4$

Answer : $n = 4$ terms

Q14. The sum of the first three terms of a G.P. is 16 and the sum of the next three terms is 128. Determine the first term, the common ratio and the sum to n terms.

Solution :

- $a + ar + ar^2 = 16$ ...(1); $ar^3 + ar^4 + ar^5 = 128$ ...(2).
- (2) $= r^3(a + ar + ar^2) = 16r^3 = 128 \Rightarrow r^3 = 8 \Rightarrow r = 2$.
- From (1): $a(1 + 2 + 4) = 16 \Rightarrow 7a = 16 \Rightarrow a = \dfrac{16}{7}$.
- $S_n = \dfrac{a(r^n - 1)}{r - 1} = \dfrac{16}{7}(2^n - 1)$

Answer : $a = \dfrac{16}{7},\ r = 2,\ S_n = \dfrac{16}{7}(2^n - 1)$

Q15. Given a G.P. with $a = 729$ and 7^{th} term 64, determine S_7.

Solution :

- $a_7 = ar^6 = 64 \Rightarrow 729\,r^6 = 64 \Rightarrow r^6 = \dfrac{64}{729} = \left(\dfrac{2}{3}\right)^6 \Rightarrow r = \dfrac{2}{3}$.
- $S_7 = \dfrac{a(1 - r^7)}{1 - r} = \dfrac{729\left(1 - (2/3)^7\right)}{1 - \frac{2}{3}} = \dfrac{729\left(1 - \frac{128}{2187}\right)}{1/3}$
- $= 2187 \times \dfrac{2059}{2187} = 2059$

Answer : $S_7 = 2059$

Q16. Find a G.P. for which the sum of the first two terms is -4 and the fifth term is 4 times the third term.

Solution :

- $a + ar = -4$ ...(1); $ar^4 = 4ar^2 \Rightarrow r^2 = 4 \Rightarrow r = \pm 2$.
- If $r = 2$: $a(1 + 2) = -4 \Rightarrow a = -\dfrac{4}{3}$. G.P.: $-\dfrac{4}{3}, -\dfrac{8}{3}, -\dfrac{16}{3}, \ldots$
- If $r = -2$: $a(1 - 2) = -4 \Rightarrow a = 4$. G.P.: $4, -8, 16, -32, \ldots$

Answer : $4, -8, 16, \ldots$ (with $r = -2$)\ or\ $-\frac{4}{3}, -\frac{8}{3}, -\frac{16}{3}, \ldots$ (with $r = 2$)

Q17. If the 4^{th}, 10^{th} and 16^{th} terms of a G.P. are x, y and z respectively, prove that x, y, z are in G.P.

Solution :

- $x = ar^3, \quad y = ar^9, \quad z = ar^{15}.$
- $\frac{y}{x} = \frac{ar^9}{ar^3} = r^6$ *and* $\frac{z}{y} = \frac{ar^{15}}{ar^9} = r^6.$
- *Since* $\frac{y}{x} = \frac{z}{y}$, *the numbers* x, y, z *are in G.P.*

Answer : x, y, z are in G.P. (common ratio r^6)

Q18. Find the sum to n terms of the sequence
$8,\ 88,\ 888,\ 8888,\ \ldots$

Solution :

- $$S_n = 8 + 88 + 888 + \cdots = \frac{8}{9}(9 + 99 + 999 + \cdots) = \frac{8}{9}\sum_{k=1}^{n}(10^k - 1)$$
- $$= \frac{8}{9}\left[\frac{10(10^n - 1)}{9} - n\right] = \frac{8}{81}(10^{\,n+1} - 9n - 10)$$

Answer : $S_n = \frac{8}{81}(10^{\,n+1} - 9n - 10)$

Q19. Find the sum of the products of the corresponding terms of the sequences $2, 4, 8, 16, 32$ and $128, 32, 8, 2, \frac{1}{2}$.

Solution :

- *Products:* $2{\cdot}128,\ 4{\cdot}32,\ 8{\cdot}8,\ 16{\cdot}2,\ 32{\cdot}\frac{1}{2} = 256, 128, 64, 32, 16.$
- *This is a G.P. with* $a = 256,\ r = \dfrac{1}{2},\ n = 5.$
- $S_5 = \dfrac{256\left(1-(1/2)^5\right)}{1-\frac{1}{2}} = 256 \cdot \dfrac{31/32}{1/2} = 496$

Answer : 496

Q20. Show that the products of the corresponding terms of the sequences $a,\ ar,\ ar^2,\ \ldots,\ ar^{\,n-1}$ and $A,\ AR,\ AR^2,\ \ldots,\ AR^{\,n-1}$ form a G.P., and find the common ratio.

Solution :

- *The* k^{th} *product* $= (ar^{k-1})(AR^{k-1}) = aA\,(rR)^{k-1}.$
- *This has first term* aA *and each term is* (rR) *times the previous one.*
- *Hence the products form a G.P. with common ratio* rR.

Answer : They form a G.P. with common ratio rR

Q21. Find four numbers forming a G.P. in which the third term is greater than the first term by 9, and the second term is greater than the 4^{th} by 18.

Solution :

- Let the numbers be $a,\ ar,\ ar^2,\ ar^3$.
- $ar^2 - a = 9 \Rightarrow a(r^2-1) = 9$ ...(1)
- $ar - ar^3 = 18 \Rightarrow -ar(r^2-1) = 18$ ...(2)
- Dividing (2) by (1): $-r = 2 \Rightarrow r = -2$. Then (1): $3a = 9 \Rightarrow a = 3$.
- Numbers: $3,\ -6,\ 12,\ -24$.

Answer : $3,\ -6,\ 12,\ -24$

Q22. If the p^{th}, q^{th} and r^{th} terms of a G.P. are a, b and c respectively, prove that $a^{q-r}\, b^{r-p}\, c^{p-q} = 1$.

Solution :

- Let first term A and ratio R: $a = AR^{p-1}$, $b = AR^{q-1}$, $c = AR^{r-1}$.
- Power of A: $(q-r)+(r-p)+(p-q) = 0$.
- Power of R: $(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q) = 0$ (on expanding).
- $\therefore\ a^{q-r}b^{r-p}c^{p-q} = A^0R^0 = 1$

Answer : $a^{q-r}b^{r-p}c^{p-q} = 1$ (proved)

Q23. If the first and the n^{th} term of a G.P. are a and b respectively, and P is the product of n terms, prove that $P^2 = (ab)^n$.

Solution :

- Terms: $a, ar, ar^2, \ldots, ar^{n-1} = b$.
- $P = a \cdot ar \cdots ar^{n-1} = a^n r^{1+2+\cdots+(n-1)} = a^n r^{\,n(n-1)/2}$.
- $P^2 = a^{2n} r^{\,n(n-1)}$. Also $(ab)^n = (a \cdot ar^{n-1})^n = (a^2 r^{n-1})^n = a^{2n} r^{\,n(n-1)}$.
- $\therefore\ P^2 = (ab)^n$

Answer : $P^2 = (ab)^n$ (proved)

Q24. Show that the ratio of the sum of the first n terms of a G.P. to the sum of terms from the $(n+1)^{th}$ to the $(2n)^{th}$ term is $\frac{1}{r^n}$.

Solution :

- Sum of first n terms: $S = \frac{a(r^n - 1)}{r - 1}$.
- Sum from $(n+1)$ to $2n$:

$$S' = S_{2n} - S_n = \frac{a(r^{2n} - 1)}{r - 1} - \frac{a(r^n - 1)}{r - 1} = \frac{a\,r^n(r^n - 1)}{r - 1}.$$

- $$\frac{S}{S'} = \frac{a(r^n - 1)/(r - 1)}{a\,r^n(r^n - 1)/(r - 1)} = \frac{1}{r^n}$$

Answer : $\frac{S}{S'} = \frac{1}{r^{\,n}}$ (proved)

Q25. If a, b, c and d are in G.P., show that $(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$.

Solution :

- Let $b=ar,\ c=ar^2,\ d=ar^3$.
- L.H.S. $=a^2(1+r^2+r^4)\cdot a^2r^2(1+r^2+r^4)=a^4r^2(1+r^2+r^4)^2$.
- R.H.S.

$$=(a^2r+a^2r^3+a^2r^5)^2=\left(a^2r(1+r^2+r^4)\right)^2=a^4r^2(1+r^2+r^4)^2.$$

- $\therefore$ L.H.S. $=$ R.H.S.

Answer : Identity proved

Q26. Insert two numbers between 3 and 81 so that the resulting sequence is a G.P.

Solution :

- Let the G.P. be $3,\ G_1,\ G_2,\ 81$ (four terms), $a=3$.
- $ar^3=81\Rightarrow 3r^3=81\Rightarrow r^3=27\Rightarrow r=3$.
- $G_1=3\times3=9,\quad G_2=9\times3=27$.

Answer : The two numbers are 9 and 27

Q27. Find the value of n so that $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be the geometric mean between a and b.

Solution :

- G.M. of a, b is $\sqrt{ab} = a^{1/2}b^{1/2}$. Set $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n} = a^{1/2}b^{1/2}$.
- $a^{n+1}+b^{n+1} = a^{1/2}b^{1/2}(a^n+b^n) = a^{n+1/2}b^{1/2} + a^{1/2}b^{n+1/2}$.
- $a^{n+1/2}\left(a^{1/2}-b^{1/2}\right) = b^{n+1/2}\left(a^{1/2}-b^{1/2}\right) \Rightarrow \left(\dfrac{a}{b}\right)^{n+1/2} = 1$.
- $\therefore\ n+\dfrac{1}{2} = 0 \Rightarrow n = -\dfrac{1}{2}$

Answer : $n = -\dfrac{1}{2}$

Q28. The sum of two numbers is 6 times their geometric mean. Show that the numbers are in the ratio $\left(3+2\sqrt{2}\right) : \left(3-2\sqrt{2}\right)$.

Solution :

- Let the numbers be a, b. Given $a+b = 6\sqrt{ab}$.
- $\dfrac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \dfrac{6\sqrt{ab}+2\sqrt{ab}}{6\sqrt{ab}-2\sqrt{ab}} = \dfrac{8}{4} = 2$, i.e. $\dfrac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})^2} = 2$.
- $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \sqrt{2} \Rightarrow \dfrac{\sqrt{a}}{\sqrt{b}} = \dfrac{\sqrt{2}+1}{\sqrt{2}-1}$ (componendo\,\&\,dividendo).
- $\dfrac{a}{b} = \dfrac{(\sqrt{2}+1)^2}{(\sqrt{2}-1)^2} = \dfrac{3+2\sqrt{2}}{3-2\sqrt{2}}$

Answer : $a : b = \left(3+2\sqrt{2}\right) : \left(3-2\sqrt{2}\right)$

Q29. If A and G are respectively the A.M. and G.M. between two positive numbers, prove that the numbers are $A \pm \sqrt{(A+G)(A-G)}$.

Solution :

- Let the numbers be a, b: $A = \dfrac{a+b}{2} \Rightarrow a+b = 2A$; $G = \sqrt{ab} \Rightarrow ab = G^2$.
- So a, b are the roots of $x^2 - 2Ax + G^2 = 0$.
- $x = \dfrac{2A \pm \sqrt{4A^2 - 4G^2}}{2} = A \pm \sqrt{A^2 - G^2} = A \pm \sqrt{(A+G)(A-G)}$

Answer : Numbers $= A \pm \sqrt{(A+G)(A-G)}$

Q30. The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria originally, how many will be present at the end of the 2^{nd} hour, 4^{th} hour and n^{th} hour?

Solution :

- The counts form a G.P. with $a = 30$, $r = 2$; count at end of the n^{th} hour $= 30 \cdot 2^n$.
- End of 2^{nd} hour: $30 \cdot 2^2 = 120$.
- End of 4^{th} hour: $30 \cdot 2^4 = 480$.
- End of n^{th} hour: $30 \cdot 2^{\,n}$.

Answer : $120, \;\; 480, \;\; 30 \cdot 2^{\,n}$

Q31. What will Rs 500 amount to in 10 years after its deposit in a bank which pays annual interest rate of 10% compounded annually?

Solution :

- Amount $= P\left(1+\dfrac{R}{100}\right)^n = 500\left(1+\dfrac{10}{100}\right)^{10} = 500\,(1.1)^{10}$.
- $(1.1)^{10} = 2.5937\ldots$
- Amount $= 500 \times 2.5937 \approx \text{Rs } 1296.87$

Answer : $\approx$ Rs 1296.87

Q32. If the A.M. and G.M. of the roots of a quadratic equation are 8 and 5 respectively, obtain the quadratic equation.

Solution :

- Let the roots be α, β: $\dfrac{\alpha+\beta}{2} = 8 \Rightarrow \alpha+\beta = 16$; $\sqrt{\alpha\beta} = 5 \Rightarrow \alpha\beta = 25$.
- Equation: $x^2 - (\alpha+\beta)x + \alpha\beta = 0$.
- $x^2 - 16x + 25 = 0$

Answer : $x^2 - 16x + 25 = 0$

Miscellaneous Exercise on Chapter 8

Q1. If f is a function satisfying $f(x+y) = f(x)\,f(y)$ for all $x, y \in \mathbb{N}$ such that $f(1) = 3$ and $\sum_{x=1}^{n} f(x) = 120$, find the value of n.

Solution :

- $f(x+y) = f(x)f(y)$ *with* $f(1) = 3$ *gives* $f(x) = 3^x$.
- $\sum_{x=1}^{n} f(x) = 3 + 3^2 + \cdots + 3^n = \dfrac{3(3^n - 1)}{3 - 1} = 120$.
- $3^n - 1 = 80 \Rightarrow 3^n = 81 = 3^4 \Rightarrow n = 4$

Answer : $n = 4$

Q2. The sum of some terms of a G.P. is 315 whose first term and common ratio are 5 and 2 respectively. Find the last term and the number of terms.

Solution :

- $S_n = \dfrac{a(r^n - 1)}{r - 1} = \dfrac{5(2^n - 1)}{2 - 1} = 5(2^n - 1) = 315$.
- $2^n - 1 = 63 \Rightarrow 2^n = 64 \Rightarrow n = 6$.
- *Last term* $= ar^{n-1} = 5 \cdot 2^5 = 160$.

Answer : Number of terms $= 6$,\ last term $= 160$

Q3. The first term of a G.P. is 1. The sum of the third term and the fifth term is 90. Find the common ratio.

Solution :

- $a = 1$. $ar^2 + ar^4 = 90 \Rightarrow r^2 + r^4 = 90$.
- Put $u = r^2$: $u^2 + u - 90 = 0 \Rightarrow (u-9)(u+10) = 0 \Rightarrow u = 9$ (as $u \geq 0$).
- $r^2 = 9 \Rightarrow r = \pm 3$

Answer : $r = \pm 3$

Q4. The sum of three numbers in G.P. is 56. If we subtract $1,\ 7,\ 21$ from these numbers in that order, we obtain an A.P. Find the numbers.

Solution :

- Let the numbers be $a,\ ar,\ ar^2$: $a(1 + r + r^2) = 56$ ...(1).
- $(a-1), (ar-7), (ar^2-21)$ in A.P.:
  $2(ar-7) = (a-1) + (ar^2-21) \Rightarrow a(r-1)^2 = 8$ ...(2).
- Divide (1) by (2): $\dfrac{1+r+r^2}{(r-1)^2} = 7 \Rightarrow 2r^2 - 5r + 2 = 0 \Rightarrow r = 2 \text{ or } \frac{1}{2}$.
- $r = 2:\ a = 8 \Rightarrow 8, 16, 32$. $(r = \frac{1}{2}$ gives $32, 16, 8.)$

Answer : The numbers are $8,\ 16,\ 32$

Q5. A G.P. consists of an even number of terms. If the sum of all the terms is 5 times the sum of the terms occupying odd places, find its common ratio.

Solution :

- Let there be $2n$ terms. $S = 5\,S_{\text{odd}}$.
- $S = \dfrac{a(r^{2n}-1)}{r-1}$ and $S_{\text{odd}} = a + ar^2 + \cdots = \dfrac{a(r^{2n}-1)}{r^2-1}$.
- $\dfrac{a(r^{2n}-1)}{r-1} = 5 \cdot \dfrac{a(r^{2n}-1)}{r^2-1} \Rightarrow \dfrac{1}{r-1} = \dfrac{5}{(r-1)(r+1)}$.
- $r+1 = 5 \Rightarrow r = 4$

Answer : $r = 4$

Q6. If $\dfrac{a+bx}{a-bx} = \dfrac{b+cx}{b-cx} = \dfrac{c+dx}{c-dx}$ $(x \neq 0)$, then show that $a,\ b,\ c,\ d$ are in G.P.

Solution :

- Take the first two equal ratios and apply componendo \& dividendo:
- $\dfrac{(a+bx)+(a-bx)}{(a+bx)-(a-bx)} = \dfrac{(b+cx)+(b-cx)}{(b+cx)-(b-cx)} \Rightarrow \dfrac{a}{bx} = \dfrac{b}{cx} \Rightarrow b^2 = ac.$
- Similarly from the last two ratios: $c^2 = bd$.
- $\dfrac{b}{a} = \dfrac{c}{b} = \dfrac{d}{c}$, hence a, b, c, d are in G.P.

Answer : $a,\ b,\ c,\ d$ are in G.P. (proved)

Q7. Let S be the sum, P the product and R the sum of reciprocals of n terms in a G.P. Prove that $P^2R^n = S^n$.

Solution :

- Terms: $a, ar, \ldots, ar^{n-1}$. $S = \dfrac{a(r^n - 1)}{r - 1}$; $P = a^n r^{\,n(n-1)/2}$.
- $R = \dfrac{1}{a} + \dfrac{1}{ar} + \cdots + \dfrac{1}{ar^{n-1}} = \dfrac{r^n - 1}{a\,r^{n-1}(r-1)} \Rightarrow \dfrac{S}{R} = a^2 r^{\,n-1}$.
- $P^2 = \left(a^n r^{n(n-1)/2}\right)^2 = a^{2n} r^{\,n(n-1)} = (a^2 r^{\,n-1})^n = \left(\dfrac{S}{R}\right)^n$.
- $\therefore\ P^2R^n = S^n$

Answer : $P^2R^n = S^n$ (proved)

Q8. If a, b, c, d are in G.P., prove that $(a^n + b^n)$, $(b^n + c^n)$, $(c^n + d^n)$ are in G.P.

Solution :

- Let $b = ar,\ c = ar^2,\ d = ar^3$.
- $a^n + b^n = a^n(1 + r^n)$; $b^n + c^n = a^n r^n (1 + r^n)$;
  $c^n + d^n = a^n r^{2n}(1 + r^n)$.
- $\dfrac{b^n + c^n}{a^n + b^n} = r^n = \dfrac{c^n + d^n}{b^n + c^n}$.
- Hence the three quantities are in G.P. with common ratio r^n.

Answer : They are in G.P. (proved)

Q9. If a, b are the roots of $x^2 - 3x + p = 0$ and c, d are the roots of $x^2 - 12x + q = 0$, where a, b, c, d form a G.P., prove that $(q+p) : (q-p) = 17 : 15$.

Solution :

- $a+b=3,\ ab=p;\ \ c+d=12,\ cd=q$. Let $b=ar,\ c=ar^2,\ d=ar^3$.
- $a(1+r)=3$ and $ar^2(1+r)=12 \Rightarrow r^2=4 \Rightarrow r=2$; then $a=1$.
- So $a=1, b=2, c=4, d=8 \Rightarrow p=ab=2,\ q=cd=32$.
- $\dfrac{q+p}{q-p}=\dfrac{34}{30}=\dfrac{17}{15}$

Answer : $(q+p):(q-p)=17:15$

Q10. The ratio of the A.M. and G.M. of two positive numbers a and b is $m : n$. Show that $a : b = \left(m+\sqrt{m^2-n^2}\right) : \left(m-\sqrt{m^2-n^2}\right)$.

Solution :

- $\dfrac{A.M.}{G.M.}=\dfrac{(a+b)/2}{\sqrt{ab}}=\dfrac{m}{n} \Rightarrow a+b=\dfrac{2m}{n}\sqrt{ab}$.
- So a, b are roots of $t^2-\dfrac{2m}{n}\sqrt{ab}\,t+ab=0$, giving $t=\sqrt{ab}\,\dfrac{m\pm\sqrt{m^2-n^2}}{n}$.
- $\dfrac{a}{b}=\dfrac{m+\sqrt{m^2-n^2}}{m-\sqrt{m^2-n^2}}$

Answer : $a:b=\left(m+\sqrt{m^2-n^2}\right):\left(m-\sqrt{m^2-n^2}\right)$

Q11. Find the sum of the following series up to n terms: (i) $5+55+555+\cdots$ (ii) $.6+.66+.666+\cdots$

Solution :

- *(i)*

$$S=\frac{5}{9}(9+99+999+\cdots)=\frac{5}{9}\sum_{k=1}^{n}(10^k-1)=\frac{5}{81}(10^{\,n+1}-9n-10)$$

- .

- *(ii)* $S=6(.1+.11+.111+\cdots)=\frac{6}{9}\sum_{k=1}^{n}(1-10^{-k})$.
- $=\frac{2}{3}\left[n-\frac{1-10^{-n}}{9}\right]=\frac{2n}{3}-\frac{2}{27}(1-10^{-n})$

Answer : (i) $\frac{5}{81}(10^{\,n+1}-9n-10)$;\ \ (ii) $\frac{2n}{3}-\frac{2}{27}(1-10^{-n})$

Q12. Find the 20^{th} term of the series $2\times4+4\times6+6\times8+\cdots+n$ terms.

Solution :

- *The n^{th} term* $=(2n)(2n+2)=4n(n+1)$.
- $a_{20}=4(20)(21)=1680$

Answer : $a_{20}=1680$

Q13. A farmer buys a used tractor for Rs 12000. He pays Rs 6000 cash and agrees to pay the balance in annual instalments of Rs 500 plus 12% interest on the unpaid amount. How much will the tractor cost him?

Solution :

- *Balance* $= 6000$, *paid as* 12 *instalments of Rs* 500.
- *Interest is* 12% *of the unpaid amounts* $6000, 5500, \ldots, 500$ *(an A.P. of* 12 *terms).*
- *Sum of unpaid amounts* $= \frac{12}{2}(6000 + 500) = 6 \times 6500 = 39000$; *interest* $= 12\% \times 39000 = 4680$.
- *Total cost* $= 12000 + 4680 = \text{Rs } 16680$

Answer : Rs 16680

Q14. Shamshad Ali buys a scooter for Rs 22000. He pays Rs 4000 cash and agrees to pay the balance in annual instalments of Rs 1000 plus 10% interest on the unpaid amount. How much will the scooter cost him?

Solution :

- *Balance* $= 18000$, *paid as* 18 *instalments of Rs* 1000.
- *Unpaid amounts:* $18000, 17000, \ldots, 1000$; *sum* $= \frac{18}{2}(18000 + 1000) = 9 \times 19000 = 171000$.
- *Interest* $= 10\% \times 171000 = 17100$.

- *Total cost* $= 22000 + 17100 = \text{Rs } 39100$

Answer : Rs 39100

Q15. A person writes a letter to four of his friends and asks each to copy it and mail it to four different persons, and so on. Assuming the chain is not broken and it costs 50 paise to mail one letter, find the amount spent on postage when the 8^{th} set of letters is mailed.

Solution :

- *Letters mailed in each set:* $4,\ 4^2,\ 4^3,\ \ldots,\ 4^8$ *(a G.P.).*
- *Total letters* $= \dfrac{4(4^8-1)}{4-1} = \dfrac{4(65535)}{3} = 87380.$
- *Postage* $= 87380 \times \dfrac{1}{2}$ *rupee* $= \text{Rs } 43690$

Answer : Rs 43690

Q16. A man deposited Rs 10000 in a bank at 5% simple interest annually. Find the amount in the 15^{th} year since he deposited the amount, and the total amount after 20 years.

Solution :

- *Annual simple interest* $= 5\% \times 10000 = \text{Rs } 500.$
- *Amount in the* 15^{th} *year* $= 10000 + 14 \times 500 = \text{Rs } 17000.$
- *Total amount after* 20 *years* $= 10000 + 20 \times 500 = \text{Rs } 20000.$

Answer : Rs 17000 (15th year),\ \ Rs 20000 (after 20 years)

Q17. A manufacturer reckons that the value of a machine, which costs him Rs 15625, will depreciate each year by 20%. Find the estimated value at the end of 5 years.

Solution :

- Value after n years $= 15625\left(1 - \dfrac{20}{100}\right)^5 = 15625\left(\dfrac{4}{5}\right)^5$.
- $= 15625 \times \dfrac{1024}{3125} = 5 \times 1024 = \text{Rs } 5120$

Answer : Rs 5120

Q18. 150 workers were engaged to finish a job in a certain number of days. 4 workers dropped out on the second day, 4 more on the third day, and so on. It took 8 more days to finish the work. Find the number of days in which the work was completed.

Solution :

- Let the planned number of days be x; total work $= 150x$ worker-days.
- Actual work done in $(x+8)$ days: $150 + 146 + 142 + \cdots$ (A.P., $a = 150,\ d = -4,\ n = x+8$).
- $\dfrac{x+8}{2}\left[2(150) + (x+7)(-4)\right] = 150x \Rightarrow x^2 + 15x - 544 = 0$.
- $(x-17)(x+32) = 0 \Rightarrow x = 17$; work completed in $x + 8 = 25$ days.

Answer : 25 days

Concept Check — Test Your Understanding

Try each one in your own words first, then check the answer below.

Q1. What is a sequence?

Answer: An arrangement of numbers in a definite order according to some rule; formally, a function whose domain is mathbb{N} (or {1,2,...,k}).

Q2. How do a sequence and a series differ?

Answer: A sequence lists the terms $a_1, a_2, a_3, \ldots$; a series is their sum $a_1 + a_2 + a_3 + \cdot s$.

Q3. When is a sequence called a G.P.?

Answer: When the ratio of any term to its preceding term is constant. This constant is the common ratio r.

Q4. Write the n^{th} term of a G.P.

Answer: $a_n = ar^{n-1}$, where a is the first term and r the common ratio.

Q5. Give the sum of n terms of a G.P. for $r \neq 1$.

Answer: $S_n = (a(r^n - 1))/(r-1) = (a(1-r^n))/(1-r)$.

Q6. What is the geometric mean (G.M.) of two positive numbers a and b?

Answer: √(ab); the sequence a, √(ab), b is a G.P.

Q7. If three numbers are in G.P., how are they related?

Answer: The square of the middle term equals the product of the other two: $b^2 = ac$.

Q8. For two positive numbers, how do A.M. and G.M. compare?

Answer: A.M.≥ G.M., with equality only when the two numbers are equal.

Q9. How do you insert k geometric means between a and b?

Answer: Form a G.P. with k+2 terms: a, G_1, ..., G_k, b, where $r = (b/a)^{1/(k+1)}$.

Q10. What is the sum of an infinite G.P. when |r|<1?

Answer: S_infty=a/(1-r).