✦ EXERCISE 10.3 (DOT PRODUCT) ✦

1 *Find the angle between two vectors* $\vec{a}$ *and* $\vec{b}$ *with magnitudes* $\sqrt{3}$ *and* 2, *and* $\vec{a}\cdot\vec{b}=\sqrt{6}$.

SOLUTION

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\sqrt{6}}{\sqrt{3}\cdot 2} = \frac{1}{\sqrt{2}} \Rightarrow \boxed{\theta = \frac{\pi}{4}}$$

2 *Find the angle between* $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$ *and* $\vec{b}=3\hat{i}-2\hat{j}+\hat{k}$.

SOLUTION

- $\vec{a}\cdot\vec{b} = 3+4+3 = 10,\quad |\vec{a}| = |\vec{b}| = \sqrt{14}$
- $$\cos\theta = \frac{10}{\sqrt{14}\cdot\sqrt{14}} = \frac{10}{14} = \frac{5}{7} \Rightarrow \boxed{\theta = \cos^{-1}\frac{5}{7}}$$

3 *Find the projection of the vector* $\hat{i}-\hat{j}$ *on the vector* $\hat{i}+\hat{j}$.

SOLUTION

$$\text{Projection} = \frac{(\hat{i}-\hat{j})\cdot(\hat{i}+\hat{j})}{|\hat{i}+\hat{j}|} = \frac{1-1}{\sqrt{2}} = \boxed{0}$$ *(the vectors are perpendicular).*

4 *Find the projection of* $\hat{i}+3\hat{j}+7\hat{k}$ *on* $7\hat{i}-\hat{j}+8\hat{k}$.

SOLUTION

- $\vec{a}\cdot\vec{b} = 7-3+56 = 60,\quad |\vec{b}| = \sqrt{49+1+64} = \sqrt{114}$
- $$\text{Projection} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \boxed{\frac{60}{\sqrt{114}}}$$

✦ EXERCISE 10.3 (CONTD.) ✦

5 Show that each of $\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k})$, $\frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k})$, $\frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k})$ is a unit vector, and that they are mutually perpendicular.

SOLUTION

- $|\vec{a}| = \frac{1}{7}\sqrt{4+9+36} = \frac{1}{7}\sqrt{49} = 1$; similarly $|\vec{b}| = 1$, $|\vec{c}| = 1$ — all **unit vectors**.
- $\vec{a}\cdot\vec{b} = \frac{1}{49}(6-18+12) = 0$, $\vec{b}\cdot\vec{c} = \frac{1}{49}(18-12-6) = 0$, $\vec{c}\cdot\vec{a} = \frac{1}{49}(12+6-18) = 0$.
- All pairwise dot products are $0 \Rightarrow$ the three vectors are **mutually perpendicular**.

6 Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$.

SOLUTION

- $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = |\vec{a}|^2 - |\vec{b}|^2 = 8$.
- Put $|\vec{a}| = 8|\vec{b}|$: $64|\vec{b}|^2 - |\vec{b}|^2 = 8 \Rightarrow 63|\vec{b}|^2 = 8 \Rightarrow |\vec{b}|^2 = \dfrac{8}{63}$.
- $|\vec{b}| = \sqrt{\dfrac{8}{63}} = \dfrac{2\sqrt{14}}{21}$, $\quad |\vec{a}| = 8|\vec{b}| = \dfrac{16\sqrt{14}}{21}$.

7 Evaluate the product $(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b})$.

SOLUTION

$$(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b}) = 6\,\vec{a}\cdot\vec{a} + 21\,\vec{a}\cdot\vec{b} - 10\,\vec{b}\cdot\vec{a} - 35\,\vec{b}\cdot\vec{b}$$

$= \boxed{6|\vec{a}|^2 + 11\,\vec{a}\cdot\vec{b} - 35|\vec{b}|^2}$ (using $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$).

✦ EXERCISE 10.3 (CONTD.) ✦

8 *Find $|\vec{a}|$ and $|\vec{b}|$ if they have the **same magnitude**, the angle between them is 60°, and $\vec{a}\cdot\vec{b}=\frac{1}{2}$.*

SOLUTION

$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos 60^\circ=|\vec{a}|^2\cdot\frac{1}{2}=\frac{1}{2}\Rightarrow|\vec{a}|^2=1\Rightarrow\boxed{|\vec{a}|=|\vec{b}|=1}$

9 *Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=12$.*

SOLUTION

$(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=|\vec{x}|^2-|\vec{a}|^2=12$. *Since* $|\vec{a}|=1$: $|\vec{x}|^2-1=12\Rightarrow|\vec{x}|^2=13\Rightarrow\boxed{|\vec{x}|=\sqrt{13}}$

10 *If $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k}$, $\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$, $\vec{c}=3\hat{i}+\hat{j}$ are such that $\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}$, find λ.*

SOLUTION

- $\vec{a}+\lambda\vec{b}=(2-\lambda)\hat{i}+(2+2\lambda)\hat{j}+(3+\lambda)\hat{k}$.
- $(\vec{a}+\lambda\vec{b})\cdot\vec{c}=0\Rightarrow 3(2-\lambda)+1(2+2\lambda)+0=0$.
- $6-3\lambda+2+2\lambda=0\Rightarrow 8-\lambda=0\Rightarrow\boxed{\lambda=8}$

11 *Show that $|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$, for any two non-zero vectors $\vec{a},\vec{b}$.*

SOLUTION

$\left(|\vec{a}|\vec{b}+|\vec{b}|\vec{a}\right)\cdot\left(|\vec{a}|\vec{b}-|\vec{b}|\vec{a}\right)=|\vec{a}|^2|\vec{b}|^2-|\vec{b}|^2|\vec{a}|^2=0$.

*Since the dot product is 0, the two vectors are **perpendicular**.*

✦ EXERCISE 10.3 (CONTD.) ✦

12 *If $\vec{a}\cdot\vec{a}=0$ and $\vec{a}\cdot\vec{b}=0$, what can be concluded about the vector $\vec{b}$?*

SOLUTION

- $\vec{a}\cdot\vec{a}=|\vec{a}|^2=0 \Rightarrow |\vec{a}|=0 \Rightarrow \vec{a}=\vec{0}$ *(the zero vector).*
- *Then* $\vec{a}\cdot\vec{b}=\vec{0}\cdot\vec{b}=0$ *holds for **every** vector* $\vec{b}$.
- $\therefore$ $\vec{b}$ *can be **any vector**.*

13 *If $\vec{a},\vec{b},\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, find $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$.*

SOLUTION

- $|\vec{a}+\vec{b}+\vec{c}|^2=0 \Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0.$
- $1+1+1+2S=0 \Rightarrow 2S=-3 \Rightarrow \boxed{\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=-\frac{3}{2}}$

14 *If either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}\cdot\vec{b}=0$. But the converse need not be true. Justify with an example.*

SOLUTION

- *The **statement** is true: if* $\vec{a}=\vec{0}$ *or* $\vec{b}=\vec{0}$, *clearly* $\vec{a}\cdot\vec{b}=0$.
- *The **converse** — "$\vec{a}\cdot\vec{b}=0 \Rightarrow \vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$" — is* $\boxed{\text{FALSE}}$.
- ***Example:*** *take* $\vec{a}=\hat{i}$, $\vec{b}=\hat{j}$. *Both are non-zero, yet* $\vec{a}\cdot\vec{b}=\hat{i}\cdot\hat{j}=0$.