✦ EXERCISE 10.4 (CROSS PRODUCT) ✦

1 *Find* $|\vec{a}\times\vec{b}|$, *if* $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ *and* $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$.

SOLUTION

- $$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&-7&7\\3&-2&2\end{vmatrix}=\hat{i}(-14+14)-\hat{j}(2-21)+\hat{k}(-2+21)=19\hat{j}+19\hat{k}$$
- $$|\vec{a}\times\vec{b}|=\sqrt{0+19^2+19^2}=\boxed{19\sqrt{2}}$$

2 *Find a unit vector perpendicular to each of* $(\vec{a}+\vec{b})$ *and* $(\vec{a}-\vec{b})$, *where* $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$.

SOLUTION

- $\vec{a}+\vec{b}=4\hat{i}+4\hat{j},\quad \vec{a}-\vec{b}=2\hat{i}+4\hat{k}$
- $$\vec{n}=(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\4&4&0\\2&0&4\end{vmatrix}=16\hat{i}-16\hat{j}-8\hat{k},\quad |\vec{n}|=24$$
- *Required unit vectors* $=\pm\dfrac{\vec{n}}{|\vec{n}|}=\boxed{\pm\dfrac{1}{3}(2\hat{i}-2\hat{j}-\hat{k})}$

3 *If a unit vector* $\vec{a}$ *makes angles* $\frac{\pi}{3}$ *with* $\hat{i}$, $\frac{\pi}{4}$ *with* $\hat{j}$, *and an acute angle* θ *with* $\hat{k}$, *find* θ *and the components of* $\vec{a}$.

SOLUTION

- $\cos^2\frac{\pi}{3}+\cos^2\frac{\pi}{4}+\cos^2\theta=1\Rightarrow\frac{1}{4}+\frac{1}{2}+\cos^2\theta=1.$
- $\cos^2\theta=\frac{1}{4}\Rightarrow\cos\theta=\frac{1}{2}$ *(acute)* $\Rightarrow\boxed{\theta=\frac{\pi}{3}}$
- *Components* $(\cos\alpha,\cos\beta,\cos\gamma)=\left(\frac{1}{2},\ \frac{1}{\sqrt{2}},\ \frac{1}{2}\right).$

✦ EXERCISE 10.4 (CONTD.) ✦

4 *Show that* $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b}) = 2(\vec{a}\times\vec{b})$.

SOLUTION

- $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b}) = \vec{a}\times\vec{a}+\vec{a}\times\vec{b}-\vec{b}\times\vec{a}-\vec{b}\times\vec{b}$
- $= \vec{0}+\vec{a}\times\vec{b}+\vec{a}\times\vec{b}-\vec{0} = 2(\vec{a}\times\vec{b})$ *(using* $\vec{a}\times\vec{a}=\vec{0}$ *and* $-\vec{b}\times\vec{a}=\vec{a}\times\vec{b}$*).*

5 *Find* λ *and* μ *if* $(2\hat{i}+6\hat{j}+27\hat{k})\times(\hat{i}+\lambda\hat{j}+\mu\hat{k}) = \vec{0}$.

SOLUTION

- *The cross product is* $\vec{0} \Rightarrow$ *the vectors are collinear* $\Rightarrow$ *components are proportional:* $\dfrac{2}{1}=\dfrac{6}{\lambda}=\dfrac{27}{\mu}$.
- $\dfrac{2}{1}=\dfrac{6}{\lambda}\Rightarrow\lambda=3$; $\quad \dfrac{2}{1}=\dfrac{27}{\mu}\Rightarrow \boxed{\lambda=3,\ \mu=\dfrac{27}{2}}$

6 *Given that* $\vec{a}\cdot\vec{b}=0$ *and* $\vec{a}\times\vec{b}=\vec{0}$. *What can you conclude about* $\vec{a}$ *and* $\vec{b}$*?*

SOLUTION

- $\vec{a}\cdot\vec{b}=0\Rightarrow|\vec{a}||\vec{b}|\cos\theta=0$; $\quad \vec{a}\times\vec{b}=\vec{0}\Rightarrow|\vec{a}||\vec{b}|\sin\theta=0$.
- *For non-zero* $\vec{a},\vec{b}$ *we cannot have* $\cos\theta=0$ *and* $\sin\theta=0$ *together.*
- $\therefore$ $\boxed{|\vec{a}|=0 \text{ or } |\vec{b}|=0}$ — *at least one of them is the zero vector.*

✦ EXERCISE 10.4 (CONTD.) ✦

7 Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k},\ \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k},\ \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. Show that $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

SOLUTION

- $$\vec{a} \times (\vec{b} + \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix}$$
- Splitting the third row (property of determinants) gives $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$
- $= \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$. Hence proved (distributive law).

8 If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a} \times \vec{b} = \vec{0}$. Is the converse true? Justify with an example.

SOLUTION

- The converse — "$\vec{a} \times \vec{b} = \vec{0} \Rightarrow \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$" — is **NOT true**.
- $\vec{a} \times \vec{b} = \vec{0}$ also holds when $\vec{a}$ and $\vec{b}$ are **collinear** (parallel).
- **Example:** $\vec{a} = \hat{i},\ \vec{b} = 5\hat{i}$ are both non-zero, yet $\vec{a} \times \vec{b} = \hat{i} \times 5\hat{i} = 5(\hat{i} \times \hat{i}) = \vec{0}$.

9 Find the area of the triangle with vertices $A(1, 1, 2),\ B(2, 3, 5),\ C(1, 5, 5)$.

SOLUTION

- $\overrightarrow{BA} = -\hat{i} - 2\hat{j} - 3\hat{k}, \quad \overrightarrow{BC} = -\hat{i} + 2\hat{j}$
- $$\overrightarrow{BA} \times \overrightarrow{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -2 & -3 \\ -1 & 2 & 0 \end{vmatrix} = 6\hat{i} + 3\hat{j} - 4\hat{k}$$
- Area $= \frac{1}{2}|\overrightarrow{BA} \times \overrightarrow{BC}| = \frac{1}{2}\sqrt{36 + 9 + 16} = \boxed{\frac{\sqrt{61}}{2}\ \text{sq. units}}$

✦ EXERCISE 10.4 (CONTD.) ✦

10 Find the area of the parallelogram whose adjacent sides are $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.

SOLUTION

- $$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix} = 20\hat{i} + 5\hat{j} - 5\hat{k}$$
- Area $= |\vec{a} \times \vec{b}| = \sqrt{400 + 25 + 25} = \sqrt{450} = \boxed{15\sqrt{2} \text{ sq. units}}$

11 Let $\vec{a}, \vec{b}$ be such that $|\vec{a}| = 3$ and $|\vec{b}| = \frac{\sqrt{2}}{3}$. Then $\vec{a} \times \vec{b}$ is a unit vector if the angle between $\vec{a}$ and $\vec{b}$ is (A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$.

SOLUTION

- $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta = 1 \Rightarrow 3 \cdot \frac{\sqrt{2}}{3} \cdot \sin\theta = 1 \Rightarrow \sqrt{2}\sin\theta = 1.$
- $\sin\theta = \dfrac{1}{\sqrt{2}} \Rightarrow \theta = \dfrac{\pi}{4} \Rightarrow$ $\boxed{\text{Option (B)}}$

12 Area of a rectangle with vertices $-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$, $\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$, $\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$, $-\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$ is (A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 4.

SOLUTION

- Adjacent sides: $\vec{a} = \overrightarrow{AB} = 2\hat{i}, \quad \vec{b} = \overrightarrow{AD} = -\hat{j}.$
- $\vec{a} \times \vec{b} = (2\hat{i}) \times (-\hat{j}) = -2\hat{k} \Rightarrow$ Area $= |\vec{a} \times \vec{b}| = 2.$
- $\boxed{\text{Option (C) } 2}$